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**Confidence Levels for the Sample Mean and
Standard Deviation of a Rayleigh Process**

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Abstract

The number of independent samples necessary to characterize the parameters of a Rayleigh-distributed process within arbitrary confidence limits is derived. Stationarity of the process is assumed, along with convergence of the Central Limit theorem with regard to the probability density function of the sample mean. Perfect measurement ability is also assumed. A graph of sample size for a range of confidence coefficients and error limits is presented.

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Confidence Levels for the Sample Mean and Standard Deviation of a Rayleigh Process

1. INTRODUCTION

The Rayleigh probability density function is encountered frequently in communications and radio astronomy detection problems. For example, it characterizes the voltage at the output of an envelope detector when the input to the detector is a narrowband gaussian noise. The narrowband gaussian noise could be the result of the band-limiting characteristic of an IF strip when a broadband gaussian process is under observation. It is common to estimate the true mean value and standard deviation of such a process from samples of the continuous data taken at selected discrete instants of time.

In practice it is necessary to decide the number (n) of independent samples that must be taken before, for instance, an estimate of the mean value calculated from these samples is within a specific percentage error of the true mean value of the process. This can never be done with absolute certainty but must have associated with it a certain confidence level denoting the probability with which the mean value calculated from the n samples will fall within the error limit specified.

The number of samples necessary for the specific case of a 99% confidence level and a $\pm 1\%$ error limit is calculated in Section 2. The result is extended to the general case of a confidence level of $c\%$ and an error of $\pm e\%$, and a plot of the extended function presented graphically.

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The true mean value completely characterizes a Rayleigh probability density function. The standard deviation is proportional to this value. Moreover, if the measured mean value is within certain error limits of the true mean value, it can be shown that a "sample" standard deviation calculated from the measured mean value is within identical errors limits of the true standard deviation of the process.

2. ESTIMATION OF MEAN VALUE

The mean value of n samples of any one of the infinite number of sample functions is called the sample mean, defined by

$$\bar{y} = \frac{\sum_{i=1}^n y_i}{n} . \quad (1)$$

The y_i , the values of the n samples, have probability density functions $p(y_i)$, which are given as Rayleigh density functions. The assumption of stationarity assures us that the $p(y_i)$ are equal. Since \bar{y} is the mean value of n random variables, it too is a random variable. The probability density function for the mean value of a large number n of the y_i will, by the Central Limit theorem, converge as $n \rightarrow \infty$ on a normal or gaussian density function characterized by

$$p(\bar{y}) = \frac{1}{\sqrt{2\pi \text{Var}(\bar{y})}} \exp \left\{ -\frac{[\bar{y} - E(\bar{y})]^2}{2 \text{Var}(\bar{y})} \right\} , \quad n \rightarrow \infty . \quad (2)$$

It can be readily shown that

$$E(\bar{y}) = E(y_i)$$

and

$$\text{Var}(\bar{y}) = \frac{\text{Var}(y_i)}{n} ,$$

or,

$$\text{SD}(\bar{y}) = \frac{\text{SD}(y_i)}{\sqrt{n}} .$$

The probability density function of \bar{y} can thus be approximated by

$$p(\bar{y}) = \frac{1}{\sqrt{2\pi \frac{\text{Var}(y_i)}{n}}} \exp \left\{ -\frac{[\bar{y} - E(y_i)]^2}{2 \frac{\text{Var}(y_i)}{n}} \right\} . \quad (3)$$

If we introduce

$$\mu = \frac{\bar{y} - E(y_i)}{\frac{SD(y_i)}{\sqrt{n}}} ,$$

a change of variable could show that

$$p(\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu^2}{2}\right) . \quad (4)$$

This is of course the standard tabulated normal probability density function. From the tables it can be seen that 99% of the area (or, the probability is .99 that μ) will fall between the values of ± 2.576 . Therefore,

$$\Pr\{-2.576 < \mu < 2.576\} = .99 ,$$

or,

$$\Pr\left\{-2.576 < \frac{\bar{y} - E(y_i)}{\frac{SD(y_i)}{\sqrt{n}}} < 2.576\right\} = .99 . \quad (5)$$

This can be rearranged into the form:

$$\Pr\left\{\bar{y} - 2.576 \frac{SD(y_i)}{\sqrt{n}} < E(y_i) < \bar{y} + 2.576 \frac{SD(y_i)}{\sqrt{n}}\right\} = .99 . \quad (6)$$

If we want to assure ourselves with 99% confidence that the sample mean falls within $\pm 1\%$ of the true mean, we set

$$\Pr\{.99 E(y_i) < \bar{y} < 1.01 E(y_i)\} = .99 , \quad (7)$$

which can be rearranged into the form:

$$\Pr\{\bar{y} - .01 E(y_i) < E(y_i) < \bar{y} + .01 E(y_i)\} = .99 . \quad (8)$$

By equating the parts of Eq. (6) and (8) we find

$$.01 E(y_i) = 2.576 \frac{SD(y_i)}{\sqrt{n}} ,$$

or,

$$n = \frac{(2.576)^2 \text{Var}(y_i)}{(.01)^2 [E(y_i)]^2} . \quad (9)$$

For a process characterized by a Rayleigh density function,

$$p(y_i) = \begin{cases} \frac{y_i}{a^2} \exp\left(-\frac{y_i^2}{2a^2}\right) , & 0 < y_i < \infty ; \\ 0 , & -\infty < y_i < 0 . \end{cases} \quad (10)$$

Also,

$$E(y_i) = a \sqrt{\frac{\pi}{2}} ;$$

$$\text{Var}(y_i) = 2a^2 - \frac{a^2 \pi}{2} .$$

Therefore

$$\frac{\text{Var}(y_i)}{[E(y_i)]^2} = \frac{4}{\pi} - 1 . \quad (11)$$

Also,

$$\text{SD}(y_i) = \sqrt{\frac{4}{\pi} - 1} [E(y_i)] . \quad (12)$$

Combining Eqs. (9) and (11), we have

$$n = \frac{(2.576)^2 \left(\frac{4}{\pi} - 1\right)}{(.01)^2} \cong 17,900 . \quad (13)$$

Thus, for $n \cong 17,900$, the sample mean will fall within $\pm 1\%$ of the true mean with a 99% confidence level.

This result can of course be extended to the case of a $c\%$ confidence level and $\pm e\%$ error. Let $\pm m$ denote the values of the abscissa on the gaussian distribution between which the normalized variable μ falls in order to give $c\%$ confidence level or probability $c/100$. Then

$$\text{Pr} \{-m < \mu < m\} = \frac{c}{100} .$$

If the error is allowed to be $\pm e\%$, then Eq. (13) becomes

$$n = \frac{(m^2) \left(\frac{4}{\pi} - 1\right)}{\left(\frac{e}{100}\right)^2} . \quad (14)$$

Equation (14) is plotted in Figure 1 for values of c between 1% and 50% and values of e from $\pm 1\%$ to approximately $\pm 10\%$.

3. ESTIMATION OF STANDARD DEVIATIONS

Equation (12) defines a relationship between the true mean value and true standard deviation of a Rayleigh process. It is now shown that a "sample" standard deviation $\text{SSD}(y)$, defined in Eq. (15), will be within definable error limits of the

true process standard deviation.

Let

$$\text{SSD}(y) = \sqrt{\frac{4}{\pi} - 1} (\bar{y}) . \quad (15)$$

The extension of Eq. (8) to the general case is

$$\Pr \left\{ y - \frac{e E(y_i)}{100} < E(y_i) < \bar{y} + \frac{e E(y_i)}{100} \right\} = \frac{c}{100} . \quad (16)$$

Substituting Eq. (12) into the left side of Eq. (16) and rearranging terms yields

$$\Pr \left\{ \frac{1-e}{100} \text{SD}(y_i) < \sqrt{\frac{4}{\pi} - 1} \bar{y} < \frac{1+e}{100} \text{SD}(y_i) \right\} = \frac{c}{100} ,$$

or,

$$\Pr \left\{ \frac{1-e}{100} \text{SD}(y_i) < \text{SSD}(y) < \frac{1+e}{100} \text{SD}(y_i) \right\} = \frac{c}{100} . \quad (17)$$

In other words, if we multiply the measured sample mean by the factor

$$\left(\sqrt{\frac{4}{\pi} - 1} \approx .5227 \right)$$

we are assured that the "sample" standard deviation thus calculated is within the same error bounds of the true standard deviation as exist for the sample mean and the true mean. These error limits also exist under the same confidence conditions.

4. CONCLUSIONS

With 17,900 independent samples of a process known to be Rayleigh-distributed, we can be 99% confident that the measured-sample mean is within $\pm 1\%$ of the true mean. Similarly, 17,900 independent samples will assure that the standard deviation calculated from the sample mean is within $\pm 1\%$ of the true standard deviation. An extension of this conclusion to the general case is possible and can be obtained from Eq. (14).

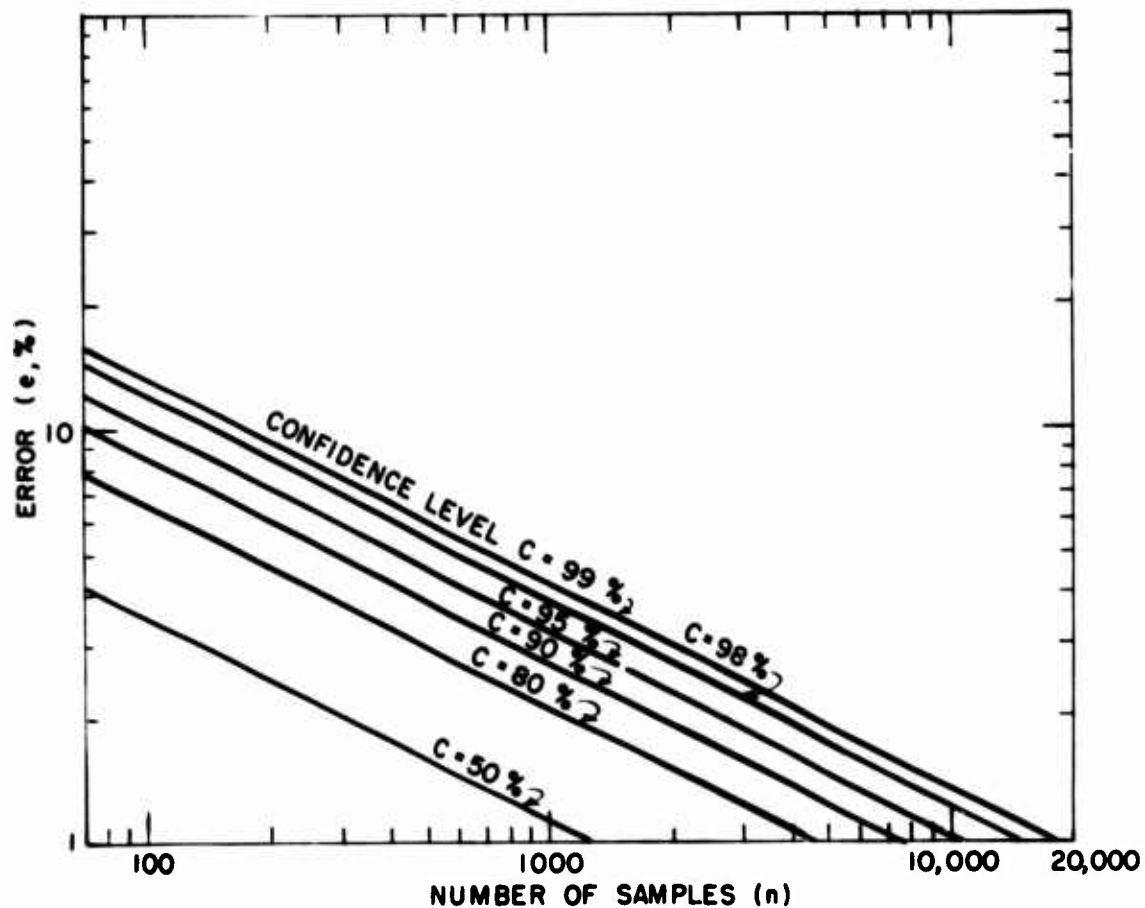


Figure 1. Error vs Number of Samples for Various Confidence Levels for the Sample Mean of a Rayleigh Process

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